Chaos assisted instanton tunneling in one dimensional perturbed periodic potential

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For the system with one-dimensional spatially periodic potential we demonstrate that small periodic in time perturbation results in appearance of chaotic instanton solutions. We estimate parameter of local instability, width of stochastic layer and correlator for perturbed instanton solutions. Application of the instanton technique enables to calculate the amplitude of the tunneling, the form of the spectrum and the lower bound for width of the ground quasi-energy zone.

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INTRODUCTION

Tunneling as inherently quantum phenomenon attracts much attention [1]. Its connection with classical chaos in semiclassical regime has also been discussed [2, 3]. A number of works were devoted to semiclassical chaos assisted tunneling between symmetry related KAM-tori in systems with mixed dynamics (well developed chaotic region coexists in phase space with regular islands) [4, 5, 6]. To describe chaos assisted tunneling in systems with mixed dynamics multi-level model Hamiltonian, primarily proposed in [2], is often used [7]. Less attention has been payed to semiclassical tunneling in KAM systems (chaotic region is not widespread) [8]. Another way to describe semiclassical tunneling is based on solutions of Hamilton equations in imaginary time and path integral formalism [1]. Instanton technique [9] was used in a very few works [10].

In this work we consider one-dimensional quantum system with periodic in space potential affected by small periodic in time perturbation. We use methods created to describe chaos in classical Hamiltonian systems to investigate essentially quantum phenomenon of tunneling. It is achieved in the framework of instanton technique, where solutions of Euclidian equations of motion (instantons) play dominating role, by the use of standard methods from the viewpoint of chaos [11]. For the systems with periodic in time perturbation energy is no more an exact integral of motion and the language of quasi-energies is more adequate [12]. For some estimations energy as an adiabatic invariant can also be used [13]. We study properties of chaotic instanton solutions and calculate the form of the spectrum and the lower bound for the width of the ground quasi-energy zone.

Hamiltonian of the system under consideration is taken in the form

$$\tilde{H} = \frac{1}{2}\tilde{p}^2 + \omega_0^2 \cos x - \epsilon x \sum_{n=-\infty}^{+\infty} \delta(t - n\tilde{T}), \quad (1)$$

$\tilde{T}$ is the real time period of perturbation, ϵ describes the strength of perturbation. The mass of the particle equals unit. Considered cosine potential corresponds nonlinear oscillator. Phase space of nonlinear oscillator has the topology of cylinder (points $(q, \tilde{p})$ and $(q + 2\pi, \tilde{p})$ are identified). Thus its (quasi-)energy spectrum is discrete. Chaos assisted tunneling between two major resonance islands (inside single potential well) for driven nonlinear oscillator has been studied numerically in [4]. In this work we study the system with x varying from $-\infty$ to $+\infty$. This results in band structure of the (quasi-)energy spectrum [14]. Systems with spatially periodic potential were studied in the instanton physics [15]. Perturbation used in (1) was exploited in the systems exhibiting quantum chaos [16, 17].

There are papers devoted chaos assisted tunneling where some analytical predictions for billiard systems basing on the path integral formulation of quantum mechanics are made [3]. Distinguishing feature of our work is analytical predictions for the system with smooth potential exploring for this purpose instanton technique adopted from quantum field theory [9].

ANALYSIS OF CHAOTIC INSTANTON SOLUTIONS

For applying instanton technique we consider solutions of classical equations of motion in *imaginary* (Euclidian) time. Hamiltonian (1) has the same form (translated on π) in Euclidian time as in real one.

Euclidian Hamiltonian of the system is $H = H_0 + V$, where

$$H_0 = \frac{1}{2}p^2 - \omega_0^2 \cos x, \quad (2)$$

and

$$V = \alpha T x \sum_{n=-\infty}^{+\infty} \delta(\tau - nT). \quad (3)$$

Here H_0 is nonperturbed Euclidian Hamiltonian of the system and V is the Euclidian potential of the perturbation. We also introduced coupling constant $\alpha \ll 1$ instead of $\epsilon \equiv \alpha T$ in order to simplify formulas.

Nonperturbed instanton solution describes the motion on the separatrix of the Hamiltonian (2). It is well known that this separatrix is destroyed by any periodic perturbation and on its place stochastic layer is appeared [13]. Perturbed instanton solutions correspond to the motion in vicinity of the separatrix inside the layer. Therefore instead of one instanton solution connecting neighbor maxima of nonperturbed Euclidian potential (classical vacuum states in real time potential) we obtain a manyfold of instanton solutions of Euclidian equations placed inside the stochastic layer.

We calculate parameter of local instability, width of stochastic layer and correlator for perturbed instanton solutions. It is convenient to describe dynamics of the system in action-angle variables [11]. Equation of motion for action variable has the form

$$\dot{I} = -\frac{\alpha \dot{x}}{\omega} \left(2 \sum_{m=1}^{+\infty} \cos\left(m\nu\tau\right) + 1 \right). \qquad (4)$$

Here $\omega(I) \equiv dH_0/dI$ is the nonlinear frequency [11]. Instead of angle variable we introduce phase of external force ψ defined by the relation $\dot{\psi} = \nu \equiv 2\pi/T$ [13]. Let $H_s \equiv \omega_0^2$ to denote the energy of nonperturbed system on the separatrix. Continuous equations of motion for I and ψ can be reduced to discrete mapping for the phase of external force in the vicinity of separatrix ($|H - H_s| \ll 1$) [13, 16, 17]

$$\psi_{n+1} = \psi_n + B_n + K_0 \sin\psi_n, \qquad (5)$$

where

$$K_0 = \frac{8\pi\alpha\nu}{\omega_0} \frac{e^{-\pi\nu/2\omega_0}}{|H - H_s|},$$

B_n are some functions of H which exact form is not essential for our purposes. We assume following [13] that due to small value of perturbation the energy practically does not change with time and equals the energy of the nonperturbed system. The map (5) with arbitrary parameter K_0 was studied by many authors, for instance [16]. Particulary, it is known that at $K_0 > 1$ motion is locally unstable and chaotic, whereas at $K_0 \leq 1$ it is stable and regular. Thus K_0 is the parameter of local instability. Condition $K_0 \sim 1$ enables us to calculate the width of stochastic layer

$$|H_s - H_b| = \frac{8\pi\alpha\nu}{\omega_0} e^{-\pi\nu/2\omega_0}, \qquad (6)$$

here H_b is the energy value on the bound of stochastic layer.

To calculate correlator for perturbed instanton solutions we use standard technique [18]. For the map (5) correlator is

$$\begin{aligned} R(\tau, \tau_0) \quad &= \frac{1}{2\pi} \int_0^{2\pi} d\psi_0 \exp\left\{ i(\psi(\tau) - \psi_0) \right\} \\ &\sim \exp\left(-\frac{\tau - \tau_0}{\tau_R} \right), \end{aligned} \qquad (7)$$

here $\psi_0 \equiv \psi(\tau_0)$ and the time of correlations decay is $\tau_R = 2\pi/(\omega \ln K_0)$. Exponential decrease of correlator shows that dynamics of the instanton solutions inside the stochastic layer ($K_0 > 1$) possesses the property of mixing (chaos) [11].

Note that perturbed one-instanton solution due to stochastic layer connects not only neighbor vacua of real time potential but also two *arbitrary* chosen vacua. Remind that to describe tunneling between non-neighbor vacua of nonperturbed system one has to take into account contribution of multi-instanton configurations [15].

THE CALCULATION OF THE TUNNELING AMPLITUDE AND GROUND ZONE WIDTH

Let us consider the tunneling between neighbor vacua (from $x \approx -\pi$ to $x \approx \pi$ for distinctness) in presence of perturbation (3). In Euclidian time this tunneling process for the nonperturbed system is described by the solution of Euclidian equations of motion with asymptotes $x = -\pi$, $p = 0$ at $\tau = -\infty$ and $x = \pi$, $p = 0$ at $\tau = +\infty$. There is only one solution satisfying these conditions for nonperturbed system (2) (one-instanton solution)

$$x_0^{inst}(\tau - \tau_0) = -\pi + 4\arctan e^{\omega_0(\tau - \tau_0)}. \qquad (8)$$

Its Euclidian action is $S^{inst} = 8\omega_0$. The instanton's position is denoted by τ_0. Due to Euclidian equations of motion and anti-symmetry of x_0^{inst} when time is inverted with respect to the point τ_0 perturbation (3) does not change the Euclidian action of the one-instanton solution (8) in the first order on the coupling constant $S_{pert}^{inst} = S^{inst} + O(\alpha^2)$. The only manifestation of the perturbation in this approximation is the appearance of a number of the additional solutions of Euclidian equations of motion with energies close to the energy of nonperturbed one-instanton solution and placed inside the stochastic layer.

Let us consider firstly nonperturbed system at arbitrary energy $-\omega_0^2 + \varepsilon$, $0 < \varepsilon < 2\omega_0^2$. One half of truncated instanton action can be easily calculated

$$S[x^{inst}(\tau, \varepsilon)] = \int_{-a(\varepsilon)}^{a(\varepsilon)} \sqrt{2(\omega_0^2 \cos x - (-\omega_0^2 + \varepsilon))}\, dx =$$

$$= 4\sqrt{4\omega_0^2 - 2\varepsilon}\, E\left(a(\varepsilon), \frac{1}{1 - \frac{\varepsilon}{2\omega_0^2}} \right), \qquad (9)$$

where $\pm a(\varepsilon) = \pm\arcsin\sqrt{1 - \frac{\varepsilon}{2\omega_0^2}}$ are turning points, function E is the elliptic integral of the second kind.

Then tunneling amplitude in *perturbed* system can be found by integration over energy of the tunneling amplitude in *nonperturbed* system with the action (9)

$$A = \int\limits_0^{\Delta H} d\varepsilon \int\limits_{x(\tau)\approx-\pi}^{x(\tau)\approx\pi} Dx \exp\left[-S[x^{inst}(\tau,\varepsilon)]\right], \qquad (10)$$

where $\Delta H = 2|H_s - H_b|$ is the stochastic layer width. The contribution of the chaotic instanton solutions is taken into account by means of integration over ε. Expression (10) shows that the probability of tunneling (square of the absolute value of the tunneling amplitude) grows while chaotic region spreads (ΔH increases).

The result is obtained in the first order on coupling constant α and does not take into account possible structure of stochastic layer is valid if the layer is narrow and is in agreement with results of numerical [20] and real [6] experiments for similar problems. We have also correspondence in (10) with the nonperturbed case [15]. Namely, if $\alpha = 0$ then $\Delta H = 0$ and the single solution describing the motion on the separatrix (nonperturbed one-instanton solution) contributes to the tunneling amplitude.

Formula (10) can be made more transparent if we use the approximate form of action (9) at $\varepsilon < 2\omega_0^2$

$$S[x^{inst}(\tau,\varepsilon)] \approx 8\omega_0 - \frac{\pi\varepsilon}{\omega_0}. \qquad (11)$$

Then in Gauss approximation we obtain the following expression for the tunneling amplitude

$$A = \int\limits_0^{\Delta H} d\varepsilon \int\limits_{-\infty}^{+\infty} dc_0 \sqrt{S[x^{inst}(\tau,\varepsilon)]} \exp\left(-S[x^{inst}(\tau,\varepsilon)]\right) \approx e^{-S^{inst}}\sqrt{S^{inst}}\Gamma F = \sqrt{8\omega_0}\Gamma e^{-8\omega_0}e^{\frac{\pi\Delta H}{\omega_0}}. \qquad (12)$$

where integration over c_0 gives the contribution of zero modes, Γ is a time of the tunneling. Formula (12) up to the factor i has the same form in real Minkovski time.

Expression (12) can be interpreted in the following way. Factor $F = \exp\left(\frac{\pi\Delta H}{\omega_0}\right) > 1$ in (12) differs perturbed and nonperturbed amplitudes and includes the contribution of the layer. One can think about F as a number of instanton solutions inside the stochastic layer.

We can find the minimal number of instanton solutions inside the stochastic layer. One can not distinguish instanton solutions within energy interval $\Delta E \sim 1/\Delta\tau$ (Heisenberg uncertainty relation). Here $\Delta\tau \sim \omega_0^{-1}$ denotes the time interval of observation. Thus the energy interval between neighbor instantons is $\Delta\varepsilon \sim \omega_0$. Therefore parameter F can be found as follows

$$F \sim 1 + \frac{\Delta H}{\Delta\varepsilon} \approx e^{\frac{\Delta H}{\omega_0}}, \qquad (13)$$

where unit takes into account the nonperturbed separatrix.

The form of the spectrum of the lower quasi-energy zone is obtained using the amplitude (12) by means of standard technique (multi-instanton contributions are taken into account) [15]:

$$E_\theta \approx \frac{1}{2}\omega_0 - 2e^{-S^{inst}}\sqrt{S^{inst}}F\cos\theta, \qquad (14)$$

here continuous variable θ parametrizes levels of the ground quasi-energy zone. Zone width

$$\Delta E \approx 4e^{-S^{inst}}\sqrt{S^{inst}}F \qquad (15)$$

differs from the case of nonperturbed case by factor F reflecting the influence of perturbation.

CONCLUSION

We applied theory of *classical* chaos for investigation of the chaos assisted *tunneling* in terms of path integral formalism in imaginary time and instanton technique. We found the parameter of local instability and the width of the stochastic layer. Exponential decrease of the correlator for any perturbed instanton solution was also demonstrated, that means it to possess the property of mixing. Then properties of the stochastic layer and classical chaotic solutions in Euclidean space (chaotic instantons) were used for the calculation tunneling amplitude, ground quasi-energy zone spectrum in the presence of the perturbation and the zone width.

General tendency for chaos assisted tunneling regime (in average – if we abstract from fluctuations) is the increase of tunneling amplitude (probability) as the strength of perturbation increases [6, 20]. It is confirmed here. The reason is the growth of the width of the chaotic layer and therefore the increase of the number of paths for particle to travel from one regular region to another. For small energies in Gauss approximation tunneling amplitude is increased by the factor $F > 1$. The life-time of the particle in the certain vacuum of the system decreases. It is connected with the widening of the (quasi-)energy zone (15).

We would like to emphasize that obtained results are not consequences of the particular choice of the non-perturbed Hamiltonian (2) or perturbation (3). Qualitatively they are valid for more general class of one-dimensional nonperturbed Hamiltonians with quadratic dependence on momentum and spatially periodic potential with single well in each period, as well as time dependence of homogeneous perturbation can be realized by any time periodic function. The reason is the universality of the separatrix destruction mechanism in these potentials affected by time-periodic perturbation [13].

Tunneling plays an important role in gauge field theories (instanton physics [9]). Experimental discovery of QCD instantons for example is an important problem [21]. Moreover it is known that classical gauge field theories are inherently chaotic [22]. Therefore the study of chaos assisted instanton tunneling in gauge field theories based on chaos criterion in quantum field theory [23] is also of essential interest.

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